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SINGULAR STRESS AT THE ENDS OF A BRITTLE CRACK FOR ZERO VALUE OF THE J INTEGRAL

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The determination of the stress and deformation in the neighborhood of a brittle defect is an important problem in the linear mechanics of fractures. The asymptotic form of the fields are superpositions of three independent types of deformation with coefficients k_1, k_2, and k_3 (normal separation, transverse shear, and longitudinal shear). It is usual to assume that these three types of deformation are necessary and sufficient to describe all possible types of behavior of a crack for the most general elastic stress distribution [1]. This assumption can also serve as the basis of experimental research on brittle fractures.

In the present paper we consider the simplest example of a boundary-value problem of plane, static elasticity for a body with a crack in which the asymptotic stress cannot be written as a superposition of a normal separation, transverse shear, and longitudinal shear. The most important feature of this stressed state is that the J integral is no longer positive-definitive, but can be either positive or negative. Analysis of the characteristics of states with $J \leq 0$, such as the opening of the crack, the surface of maximum tension, and the direction of minimum elastic energy density, illustrate that these states can occur in practice. Growth of cracks is energetically impossible in this case for any external load and surface energy. According to the Griffiths criterion, the crack becomes absolutely stable, although the stress at its ends is, as before, singular, and the displacement is discontinuous across the sides.

1. Boundary-Value Problem. We consider an unbounded plane orthotropic medium with different and purely imaginary characteristic numbers ($s_k = im_k$, $k = 1, 2$). The elastic compliance matrix a_{ij} then satisfies the condition [1]

$$\sqrt{a_{11}a_{22}} \neq a_{12} + a_{66}/2. \tag{1.1}$$

The stress, deformation, displacement, and other mechanical parameters are expressed through two analytic functions ($\Phi(z_1)$ and $\Psi(z_2)$) [1, 2]. The medium has a single discontinuity along a segment L of the real axis: $|x| \leq b$.

We assume that a part L_2 of the discontinuity with coordinates $a \leq x \leq b$ is supported by a nonstretchable membrane. The membrane therefore prevents longitudinal stretching and compression, but does not resist bending and shear. This is a somewhat complicated model of a chain that is rigid to not only stretch, but also compression. The rest of the discontinuity L is unconstrained and is denoted by $L_1 \equiv L \backslash L_2$ (see Fig. 1).

We assume a uniform stress σ_{ij}^{∞} at infinity and a continuous, symmetric, balanced load along the discontinuity. Then from the expression for the orthotropic Hooke's law [1], the boundary conditions on the contour L have the form

$$\begin{aligned} \sigma_{nn}^{\pm}(x,0) &\equiv \sigma_{22}^{\pm}(x,0) = f(x), && x \in L, \\ \sigma_{ns}^{\pm}(x,0) &\equiv \sigma_{12}^{\pm}(x,0) = h(x), && x \in L_1, \\ \sigma_{ss}^{\pm}(x,0) &\equiv \sigma_{11}^{\pm}(x,0) = g(x), && x \in L_2, \end{aligned} \tag{1.2}$$

where the unknown functions f, g, and h define the disjoining load and the initial tension of the membrane.

The solution of the boundary-value problem (1.2) is not difficult to obtain. Our assumptions (orthotropic medium, nonequal parameters m_k in the condition (1.1), and balanced load) are not essential, and are made only to simplify the resulting formulas and the interpretation of the results. As usual [2], the complex potentials of the problem are represented in the form of Cauchy integrals with unknown densities. The boundary-value problem (1.2) is then transformed into a matching problem

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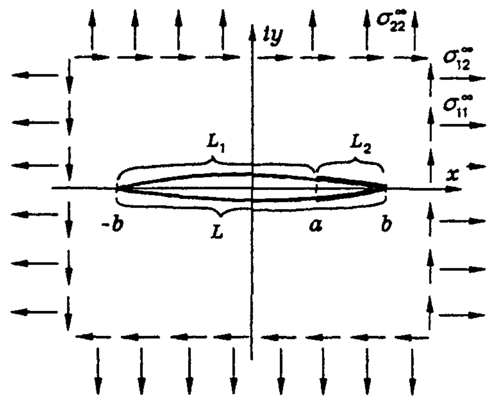

Fig. 1

on the contours L, L_1, and L_2. A unique solution of this problem is obtained from the condition that the displacements and principal force vector on the contour L must be single-valued.

The asymptotic expressions for the complex potentials near the right end $x = b$ of the discontinuity are written as

$$\Phi(z_1) = \left\{\frac{m_2k_1}{m_2 - m_1} - \frac{m_1m_2}{m_2^2 - m_1^2}G_1 + \frac{G_2}{m_2^2 - m_1^2}\right\}\frac{(z_1 - b)^{-1/2}}{2\sqrt{2}},$$
$$\Psi(z_2) = -\left\{\frac{m_1k_1}{m_2 - m_1} - \frac{m_1m_2}{m_2^2 - m_1^2}G_1 + \frac{G_2}{m_2^2 - m_1^2}\right\}\frac{(z_2 - b)^{-1/2}}{2\sqrt{2}}. \tag{1.3}$$

Here the strength coefficients k_1, G_1, and G_2 are given by

$$k_1 \doteq \frac{1}{\pi\sqrt{b}}\int_{-b}^{+b}(\sigma_{22}^{\infty} - f)\sqrt{\frac{b+x}{b-x}}\,dx, \quad G_1 = \frac{\sqrt{2}}{\pi\sqrt{b-a}}\int_a^b(\sigma_{22}^{\infty} - f)\sqrt{\frac{x-a}{b-x}}\,dx,$$
$$G_2 = \frac{\sqrt{2}}{\pi\sqrt{b-a}}\int_a^b(\sigma_{11}^{\infty} - g)\sqrt{\frac{x-a}{b-x}}\,dx. \tag{1.4}$$

We note that the tangential stresses σ_{12}^{∞} and h do not contribute to the strength coefficients for a supported end of the discontinuity. It follows from the solution that a transverse shear occurs only at the beginning of the supported part of the crack ($x = a$). At the left, unconstrained end of the crack $x = -b$ the asymptotic forms of the potentials have the usual form [1, 2]. If the support is extended over the entire length of the crack ($a = -b$), then $k_1 = G_1$ in (1.3), (1.4), and the equations below.

2. Asymptotic Stress and Displacement. Equations (1.3) and (1.4) determine the asymptotic stress and displacement distributions, which differ from the usual expressions [1, 2], and cannot be represented as superpositions of normal separation and transverse shear. For example, in the isotropic limit [2], (1.3) gives

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{bmatrix} = \frac{a_1}{4\sqrt{r}}\begin{bmatrix}4\cos\frac{\vartheta}{2} + \cos\frac{5\vartheta}{2}\\ 4\cos\frac{\vartheta}{2} - \cos\frac{5\vartheta}{2}\\ \sin\frac{5\vartheta}{2}\end{bmatrix} + \frac{a_2}{2\sqrt{r}}\begin{bmatrix}-\cos\frac{\vartheta}{2}\\ \cos\frac{\vartheta}{2}\\ -\sin\frac{\vartheta}{2}\end{bmatrix}; \tag{2.1}$$

$$u_1 + iu_2 = \frac{\sqrt{r}}{4\mu}\left\{2æa_1\exp\frac{i\vartheta}{2} - a_1\exp\frac{i3\vartheta}{2} - 2a_2\exp\left(-\frac{i\vartheta}{2}\right)\right\}; \tag{2.2}$$

$$J_1 + iJ_2 = \frac{\pi(1+æ)}{8\mu}(a_1^2 + 2a_1a_2); \tag{2.3}$$

$$a_1 = \frac{k_1}{\sqrt{2}} + \frac{1}{\sqrt{8}}(G_2 - G_1), \quad a_2 = \frac{k_1}{\sqrt{8}} - \frac{3}{4\sqrt{2}}(G_2 - G_1). \tag{2.4}$$

Here (r, ϑ) are polar coordinates with the origin at the point $x = b$; æ is the constant of the plane problem; μ is the shear modulus; J_1, J_2 are the components of the defect-moving force, where J_1 controls the energy consumed in increasing the length of the crack, and corresponds to the well-known, J-integral, while $J_2 \equiv 0$ in a coordinate system fixed to the end of the crack.

We see from (2.1)-(2.4) that the expressions for the stress and displacements are composed of the fields of a normal separation with coefficient k_1 and an independent symmetric field with strength coefficient $(G_2 - G_1)$.

3. State of Absolute Stability of the Crack. From (2.3) and (2.4) we obtain an expression for the J-integral in terms of the strength coefficients:

$$J_1 \equiv J = \frac{\pi(1+æ)}{8\mu}\left\{k_1^2 - \frac{1}{4}(G_2 - G_1)^2\right\}. \tag{3.1}$$

Therefore in our problem the J-integral is not positive definite, but can become negative. An increase in the length of the crack from the end $x = b$ of the supported part is accompanied by absorption, and not release, of energy. Therefore growth of the crack in the stressed state (2.1) is energetically impossible when $J = 0$ for any external load and surface energy, and such a state is called a state of absolute stability of the crack.

We determine the characteristic parameters of the state of absolute stability. Because the J-integral given by (2.3) is quadratic, there are two possible cases:

$$J = 0 \quad \text{when} \quad a_1 = 0 \qquad \text{or} \qquad a_1 = -2a_2. \tag{3.2}$$

1. $a_1 = 0$. The tensile stress in polar coordinates is described by the function

$$\sigma_{\vartheta\vartheta}(r, \vartheta) = \frac{a_2}{2\sqrt{r}} \cos\frac{3\vartheta}{2}, \tag{3.3}$$

whose maximum occurs at $\vartheta = 0$, i.e., along the length of the crack. The elastic energy density $W(r, \vartheta)$ near the right end of the crack is independent of the polar angle

$$W(r, \vartheta) = a_2^2/8\mu r,$$

and therefore the well-known criterion for selecting the direction of growth of the crack according to the maximum value of the function $W(r, \vartheta)$ does not yield a unique value of the polar angle. The supported part of the cracks opens without overlap of the sides:

$$u_1 + iu_2\Big|_{-\pi}^{+\pi} = \frac{ia_2}{\mu}\sqrt{r}.$$

Therefore the first case can occur in practice, since the sides do not overlap. When we leave the state of absolute stability the preferred direction of propagation of a main crack coincides with the direction of the initial defect according to the condition $\sigma_{\vartheta\vartheta}(r, \vartheta) \to \max$.

2. $a_1 = -2a_2$. The tensile stress is, in polar coordinates

$$\sigma_{\vartheta\vartheta}(r, \vartheta) = \frac{a_1}{4\sqrt{r}}\left(3\cos\frac{\vartheta}{2} - \cos\frac{3\vartheta}{2}\right),$$

and the maximum of this function occurs at $\vartheta = \pm\pi/2$.

The elastic energy density has the form

$$W(r,\vartheta) = \frac{a_1^2}{8\mu r}\{(æ - 1)(1 + \cos\vartheta) + \cos^2\vartheta\},$$

and its minimum value occurs when $2\cos\vartheta = (1 - æ) \leq 0$, i.e., for $|\vartheta| \geq \pi/2$, and depends on the value of the Poisson ratio.

The shape of the opening of the crack is determined by (2.2) and (3.2) and is given by

$$u_1 + iu_2\Big|_{-\pi}^{+\pi} = \frac{ia_1 æ}{\mu}\sqrt{r}.$$

Therefore in the second case we also have absolute stability without overlap of the sides. The preferred direction of propagation of the crack according to both criteria (minimum elastic energy and maximum tensile stress) is approximately perpendicular to the contour of the crack.

4. Correction of the Surface Energy for Deformation Constraint. We consider a crack whose supported part is small: $(b - a) << b$. The growth of such a crack can be described in the framework of the quasi-brittle model, in which the departure from the brittle mechanism is localized to a small neighborhood of the end.

The expression for the J-integral in (3.1) is made up of positive and negative terms. The latter can be interpreted as the correction to the strength parameter of the material in the Griffiths criterion, since it is localized in a small neighborhood of the end:

$$\gamma_0 = \gamma + \Delta\gamma, \quad 2\Delta\gamma = \frac{\pi(1+æ)}{32\mu}(G_2 - G_1)^2. \tag{4.1}$$

Here γ is the initial surface energy, γ_0 is the surface energy density along the discontinuity, and $\Delta\gamma$ is the correction due to constraint of the deformation, and depends on the external load and the dimensions of the end zone. In particular, if the external load is constant, then we obtain from (1.4) and (4.1).

$$\Delta\gamma = \frac{\pi(1+æ)}{128\mu}(b - a)(\sigma_{22}^{\infty} - \sigma_{11}^{\infty} + g - f)^2, \tag{4.2}$$

which shows that the tension g of the supporting membrane could theoretically be used to control the strength of the material in the sense of (4.1). Note that the disjoining force f also appears in the expression for the correction (4.2). This shows that the cohesive load has not only a normal component f, as in the Barenblatt model, but also a longitudinal component g, if longitudinal deformations are constrained in the end zone.

5. Conclusions. 1. We have considered the boundary-value problem (1.2) of plane elastostatics for a body with a crack in which the asymptotic stress cannot be represented as a superposition of the fields of normal separation and transverse shear.

2. We have obtained asymptotic expressions for the complex potentials (1.3), the stress (2.1), the displacement (2.2), and expressions for the J-integral (2.3) and (3.1).

3. In the problem considered here the J-integral is not positive definite, but can take either positive or negative values.

4. States in which $J = 0$ [see Eq. (3.2)] are called states of absolute stability of a brittle crack. For this case we have considered the shape of the opening of the end of the crack, the tensile stress $\sigma_{\vartheta\vartheta}(r, \vartheta)$, and the elastic energy density $W(r, \vartheta)$, and also we have predicted the spread of a main crack (upon overcoming the state of absolute stability) according to the well-known criteria $\sigma_{\vartheta\vartheta} \to \max$ and $W \to \min$.

5. The negative term in the expression (3.1) for the J-integral is interpreted as the correction (4.1) of the surface energy for the constraint of the deformation. Example (4.2) extends the well-known Barenblatt model of a crack. It is shown that the cohesive property has not only a normal component, but also a longitudinal component acting along the contour of the crack, if the deformations are constrained.

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